<u>Properties</u>: (1) **Linearity**: $(aX + bY)(f) = aX(f) + bX(g)$

(2) **Tensorial**: $(fX)(g) = f(X(g))$

In terms of the Euclidean coordinates $x^1, \dots, x^n$ on $\mathbb{R}^n$,

we can express any vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ as

$$X(x^1, \dots, x^n) = \big(\underbrace{a^1(x^1, \dots, x^n), \dots\dots, a^n(x^1, \dots, x^n)}_{\text{smooth functions}}\big)$$

$$= \sum_{i=1}^{n} a^i e_i \quad , \quad \{e_i\} \text{ std. basis of } \mathbb{R}^n$$

Since e_i corresponds to $\frac{\partial}{\partial x^i}$ as operators, therefore

$$\boxed{X = a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} + \cdots + a^n \frac{\partial}{\partial x^n}}$$

Since vector fields can be viewed as operators on $C^\infty(\mathbb{R}^n)$,

we can consider their compositions:

$$\begin{tikzcd} C^\infty(\mathbb{R}^n) \arrow[r, shift left, "X"] \arrow[r, shift right, "Y"'] & C^\infty(\mathbb{R}^n) \arrow[r, shift left, "Y"] \arrow[r, shift right, "X"'] & C^\infty(\mathbb{R}^n) \end{tikzcd}$$

i.e. $\quad f \longmapsto X(f) \longmapsto Y(X(f))$

or $\quad f \longmapsto Y(f) \longmapsto X(Y(f))$

Question: Does $X(Y(f)) = Y(X(f))$?

Yes for $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$ since mixed partial derivatives commute: $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$

No in general.

Def$^{\underline{n}}$: For any vector fields X, Y on $\mathbb{R}^n$, we define their Lie bracket as

$$[X,Y] := XY - YX$$

i.e. $[X,Y](f) = X(Y(f)) - Y(X(f))$.

$$\forall f \in C^\infty(\mathbb{R}^n)$$

$$\boxed{\textbf{Note: } \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] \equiv 0}$$

coordinate vector fields commute

Lemma: $[X,Y]$ is a vector field.

Proof: Write $X = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$; $Y = \sum_{j=1}^n b^j \frac{\partial}{\partial x^j}$

$$X(Y(f)) = \left(\sum_{i=1}^n a^i \frac{\partial}{\partial x^i}\right)\left(\sum_{j=1}^n b^j \frac{\partial f}{\partial x^j}\right)$$

$$= \sum_{i,j=1}^{n} a^i b^j \frac{\partial^2 f}{\partial x^i \partial x^j} + \sum_{i,j=1}^{n} a^i \frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j}$$

Similarly,

$$Y(X(f)) = \sum_{i,j=1}^{n} a^i b^j \frac{\partial^2 f}{\partial x^j \partial x^i} + \sum_{i,j=1}^{n} b^j \frac{\partial a^i}{\partial x^j} \frac{\partial f}{\partial x^i}$$

$$\Rightarrow [X,Y](f) = \sum_{i,j=1}^{n} a^i \frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j} - \sum_{i,j=1}^{n} b^j \frac{\partial a^i}{\partial x^j} \frac{\partial f}{\partial x^i}$$

i.e. $$[X,Y] = \underbrace{\sum_{i,j=1}^{n} \left(a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}}_{\text{a vector field !}}$$

$\square$

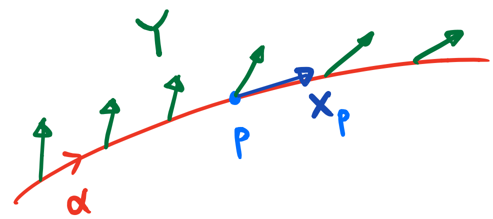
Now we recall how to take directional derivatives of a vector field in $\mathbb{R}^n$, one natural way is to differentiate component-wise:

$$D_X Y = D_X \left(\sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i} \right) := \sum_{i=1}^{n} X(a^i) \frac{\partial}{\partial x^i}.$$

Equivalently, fix a point $p \in \mathbb{R}^n$, to compute $D_X Y(p)$, we take ANY curve α s.t. $\alpha(0) = p$ and $\alpha'(0) = X_p$

$$D_X Y(p) = \left.\frac{d}{dt}\right|_{t=0} Y(\alpha(t))$$

$$= \lim_{t \to 0} \frac{Y(\alpha(t)) - Y(p)}{t}$$



<u>Properties of $D_X Y$</u>: Let X, Y, Z be vector fields on $\mathbb{R}^n$, $a, b \in \mathbb{R}$ be constants, $f \in C^\infty(\mathbb{R}^n)$. We have:

(1) Linearity in both variables:

$$D_X(aY + bZ) = a D_X Y + b D_X Z$$

$$D_{aX+bY} Z = a D_X Z + b D_Y Z$$

(2) Liebniz rule: $D_X(fY) = X(f)\, Y + f D_X Y$

(3) Tensorial: $D_{fX} Y = f D_X Y$

(4) Torsion free: $\boxed{D_X Y - D_Y X = [X, Y]}$

(5) Metric compatibility:

$$\boxed{X \langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle X, D_X Z \rangle}$$